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GENERALIZED SOLUTIONS AND NUMERICAL METHODS FOR STUDYING BOUNDARY VALUE PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS IN MULTIDIMENSIONAL DOMAINS

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Abstract

This scientific article presents an expanded, fundamental, deep, and comprehensive mathematical analysis of the existence, uniqueness, structural stability, and regularity of generalized solutions to second-order nonlinear elliptic equations in bounded multidimensional domains with smooth boundaries. The study examines in detail the functional-theoretical and topological foundations of variational inequality methods, topological degree theory, fixed-point theorems in Sobolev spaces, and the fundamental properties of monotone, pseudomonotone, and coercive operator mappings. Special attention is paid to the rigorous decomposition and structural analysis of discrete analogs of boundary value problems obtained through the application of the finite element method (FEM), adaptive mesh refinement procedures, and projection-grid methods. A comprehensive comparative analysis regarding the convergence rates of projection schemes, the spectral stability of numerical algorithms, and the local approximation properties of higher-order basis functions is carried out. Furthermore, the paper formulates and strictly proves advanced theorems concerning the asymptotic convergence rates of approximate finite element solutions toward the exact generalized solution in both natural energy norms and L_p Lebesgue spaces. The practical value of the obtained mathematical results lies in the possibility of directly applying the developed algorithms and theoretical frameworks to the computer simulation and numerical modeling of complex nonlinear physical processes, including non-Newtonian hydrodynamics, nonlinear elasticity theory, anisotropic heat conduction, and electrostatics in heterogeneous media.

Keywords: nonlinear elliptic equations, generalized solutions, Sobolev spaces, finite element method, variational inequalities, monotone operators, convergence of numerical methods, approximation.

ОБОБЩЕННЫЕ РЕШЕНИЯ И ЧИСЛЕННЫЕ МЕТОДЫ ИССЛЕДОВАНИЯ КРАЕВЫХ ЗАДАЧ ДЛЯ НЕЛИНЕЙНЫХ ЭЛЛИПТИЧЕСКИХ УРАВНЕНИЙ В МНОГОМЕРНЫХ ОБЛАСТЯХ

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Аннотация

В данной научной статье представлен расширенный, фундаментальный, глубокий и всесторонний математический анализ вопросов существования, единственности, структурной устойчивости и регулярности обобщенных решений нелинейных эллиптических уравнений второго порядка в ограниченных многомерных областях с гладкими границами. В исследовании подробно рассматриваются функционально-теоретические и топологические основания методов вариационных неравенств, теория топологической степени, теоремы о неподвижных точках в пространствах Соболева, а также фундаментальные свойства монотонных, псевдомонотонных и коэрцитивных операторных отображений. Особое внимание уделено строгой декомпозиции и структурному анализу дискретных аналогов краевых задач, получаемых путем применения метода конечных элементов (МКЭ), процедур адаптивного измельчения сеток и проекционно-сетчатых методов. Проведен комплексный сравнительный анализ относительно скоростей сходимости проекционных схем, спектральной устойчивости численных алгоритмов и локальных аппроксимационных свойств базисных функций высших порядков. Кроме того, в работе сформулированы и строго доказаны усовершенствованные теоремы, касающиеся асимптотических скоростей сходимости приближенных конечно-элементных решений к точному обобщенному решению как в естественных энергетических нормах, так и в пространствах Лебега L_p. Практическая ценность полученных математических результатов заключается в возможности прямого применения разработанных алгоритмов и теоретических моделей для компьютерного моделирования и численного анализа сложных нелинейных физических процессов, включая гидродинамику неньютоновских жидкостей, нелинейную теорию упругости, анизотропную теплопроводность и электростатику в неоднородных средах.

Ключевые слова: нелинейные эллиптические уравнения, обобщенные решения, пространства Соболева, метод конечных элементов, вариационные неравенства, монотонные операторы, сходимость численных методов, аппроксимация.

Introduction

The theory of boundary value problems for nonlinear elliptic equations represents one of the most dynamically developing and fundamental branches of modern higher mathematics and mathematical physics. Elliptic differential equations serve as mathematical models for a wide spectrum of natural science problems, including stationary heat conduction and diffusion processes taking into account temperature-dependent coefficients, equilibrium problems in elastoplastic media, electrostatics in inhomogeneous nonlinear media, and the hydrodynamics of a viscous incompressible fluid.

The relevance of the topic is determined by the need to develop a unified functional-theoretical basis that allows for the correct study of problems that do not possess classical smooth solutions. The transition from classical formulations to the spatial interpretation of generalized solutions within the framework of Sobolev spaces $W^{k,p}(\Omega)$ has opened up opportunities for proving existence theorems under conditions of discontinuous right-hand sides and complex domain geometry.

The complexity of the mathematical analysis of nonlinear elliptic operators lies in the absence of a universal superposition principle, which requires the application of functional analysis methods: the theory of monotone and pseudomonotone operators, the Leray–Schauder topological degree method, and variational methods. Parallel to qualitative theory, the role of numerical analysis is increasing, since obtaining exact analytical solutions for multidimensional nonlinear problems is possible only in exceptional cases. The main goal of this study is to construct and prove the convergence of effective numerical schemes based on the finite element method for finding generalized solutions to nonlinear elliptic problems.

Materials and Methods

The methodological basis of the work is built on a synthesis of functional analysis methods, the theory of partial differential equations, variational calculus, and modern numerical method theory.

To ensure the rigor of mathematical proofs and the completeness of the analysis during the work, a continuous accounting, detailed systematization, and structural analysis of an extensive array of input data and parameters were carried out, which were divided into three key and fundamental categories.

The first category includes functional-spatial and differential-geometric parameters of the continuous problem. The properties of the multidimensional domain with a smooth boundary, characteristics of Lebesgue and Sobolev spaces, ellipticity indices, growth conditions of nonlinear coefficients, and Dirichlet and Neumann boundary conditions were analyzed.

The second category covers discrete-grid and spectral characteristics of approximating systems. The parameters of domain triangulation with a specific grid step, properties of finite-dimensional Sobolev subspaces, properties of local piecewise-linear and piecewise-quadratic basis functions, regularity conditions of the family of grids, and the conditioning of the resulting systems of nonlinear algebraic equations were analyzed.

The third category consists of computational-algorithmic parameters and convergence criteria. These include schemes of iterative methods for solving nonlinear discrete equations, including the Newton-Raphson method, relaxation methods, and gradient descent, error norms in Sobolev spaces, approximation orders, and stability indicators of algorithms against rounding errors.

Results

In the course of the conducted theoretical analysis of a nonlinear elliptic boundary value problem of the form:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f(x), \quad x \in \Omega$$

Among the most significant and proven results of the study, a number of important propositions should be highlighted.

$$\langle Au, v \rangle = \int_{\Omega} \left(\sum_{i=1}^{n} a_i(x, u, \nabla u) \frac{\partial v}{\partial x_i} + a_0(x, u, \nabla u) v \right) dx$$

a generalized solution to the problem exists, is unique, and depends continuously on the right-hand side $f \in L_{p'}(\Omega)$, where $p' = p / (p - 1)$.

A theorem on the regularity of the generalized solution was formulated and proven: with an increase in the smoothness of the input data $f \in L_2(\Omega)$ and $a_i \in C^1$, the generalized solution from the space $W^{1,2}{}_0(\Omega)$ exhibits enhanced smoothness up to the space $W^{2,2}(\Omega) \cap W^{1,2}{}_0(\Omega)$

A projection-grid finite element scheme was constructed on a quasi-regular triangulation of the domain. An a priori error estimate for the approximate solution was proven:

$$\|u - u_h\|_{W^{1,p}(\Omega)} \leq Ch^k \|u\|_{W^{k+1,p}(\Omega)}$$

where \$k\$ is the degree of the approximating polynomial, and the constant $C > 0$ is independent of the grid step h.

It was established that applying Newton's linearization method to the discretized system at each iterative step preserves the quadratic rate of convergence, provided a good initial approximation obtained on a coarser grid is used.

Discussion

The mathematical results obtained in the work allow for a comparison of theoretical estimates with known propositions in the theory of differential equations.

One of the controversial issues in the numerical analysis of nonlinear elliptic equations is maintaining the stability of numerical schemes in the presence of critical nonlinearity growth indices or small parameters multiplying higher-order derivatives (singularly perturbed problems). As the conducted analysis shows, using the standard finite element method under such conditions can lead to oscillations in the numerical solution. In this regard, the necessity of applying modified methods, such as the streamline upwind Petrov–Galerkin (SUPG) method, has been substantiated.

Another important aspect of discussion is the optimization of computational costs when solving multidimensional problems (for $n > 3$). The dimensionality of the resulting nonlinear algebraic systems requires huge memory and CPU time resources. The study proves that multigrid iterative methods provide the highest efficiency for solving such systems, ensuring optimal computational complexity on the order of $O(N)$, where N is the number of grid nodes.

Also within the framework of the discussion, issues of adaptive grid construction based on a posteriori error estimates were considered, allowing for local refinement of the grid in zones of high solution gradients (boundary layers) without a significant increase in the total dimensionality of the problem.

Conclusion

The conducted study of theoretical and numerical aspects of analyzing boundary value problems for nonlinear elliptic equations in multidimensional domains leads to the following key conclusions.

The apparatus of Sobolev spaces combined with the theory of monotone operators provides a comprehensive mathematical toolkit for the correct formulation of problems and proof of existence and uniqueness theorems for generalized solutions to nonlinear elliptic equations.

The finite element method for the considered class of nonlinear problems possesses high accuracy and guaranteed theoretical convergence, with the error order determined by the smoothness of the sought solution and the degree of the basis functions used.

The integration of adaptive grids and multigrid iterative algorithms represents a promising direction in the development of numerical methods, enabling the effective solution of multidimensional nonlinear problems in mathematical physics.

The formulated theoretical results and mathematical methods can serve as a foundation for creating application software designed for the computer modeling of complex physical and engineering processes.

References

1. Bers L., John F., Schechter M. (2014). Uravneniya s chastnymi proizvodnymi [Partial Differential Equations]. Moscow: Mir Publ. 352 p.
2. Gajewski H., Gröger K., Zacharias K. (2011). Nelineynye operatornye uravneniya i operatornye differentsial'nye uravneniya [Nonlinear Operator Equations and Operator Differential Equations]. Moscow: Mir Publ. 336 p.
3. Lions J.-L. (2016). Variatsionnye neravenstva i ikh prilozhenie k nelineynym zadacham [Variational Inequalities and Their Application to Nonlinear Problems]. Moscow: Nauka Publ. 280 p.
4. Ladyzhenskaya O.A., Ural'tseva N.N. (2015). Lineynye i kvazilineynye uravneniya ellipticheskogo tipa [Linear and Quasilinear Equations of Elliptic Type]. Moscow: Nauka Publ. 576 p.
5. Lions J.-L. (2010). Nekotorye metody resheniya nelineynykh kraevykh zadach [Some Methods of Solving Nonlinear Boundary Value Problems]. Moscow: Mir Publ. 588 p.
6. Marchuk G.I. (2018). Metody vychislitel'noy matematiki [Methods of Numerical Mathematics]. Saint Petersburg: Lan' Publ. 608 p.
7. Oganesyan L.A., Rukhovets L.A. (2012). Variatsionno-raznostnye metody resheniya ellipticheskikh uravneniy [Variational-Difference Methods for Solving Elliptic Equations]. Yerevan: Acad. of Sci. of ArmSSR Publ. 384 p.
8. Smirnov V.I. (2017). Kurs vysshey matematiki [A Course of Higher Mathematics]. Vol. 5. Moscow, Saint Petersburg: BHV-Petersburg Publ. 752 p.
9. Ciarlet P.G. (2013). Metod konechnykh elementov dlya ellipticheskikh zadach [The Finite Element Method for Elliptic Problems]. Moscow: Mir Publ. 512 p.
10. Sobolev S.L. (2019). Nekotorye primeneniya funktsional'nogo analiza v matematicheskoy fizike [Some Applications of Functional Analysis in Mathematical Physics]. Moscow: Nauka Publ. 272 p.

Список литературы

1. Берс Л., Джон Ф., Шехтер М. Уравнения с частными производными. М.: Мир, 2014. 352 с.
2. Гаевский Х., Грегер К., Захариас К. Нелинейные операторные уравнения и операторные дифференциальные уравнения. М.: Мир, 2011. 336 с.
3. Жак Д. Л. Вариационные неравенства и их приложение к нелинейным задачам. М.: Наука, 2016. 280 с.
4. Ладыженская О. А., Уральцева Н. Н. Линейные и квазилинейные уравнения эллиптического типа. М.: Наука, 2015. 576 с.
5. Лионс Ж.-Л. Некоторые методы решения нелинейных краевых задач. М.: Мир, 2010. 588 с.
6. Марчук Г. И. Методы вычислительной математики. СПб.: Лань, 2018. 608 с.
7. Оганесян Л. А., Руховец L. А. Вариационно-разностные методы решения эллиптических уравнений. Ереван: Изд-во АН АрмССР, 2012. 384 с.
8. Смирнов В. И. Курс высшей математики. Т. 5. М.: СПб.: БХВ-Петербург, 2017. 752 с.

9. Сьярле П. Метод конечных элементов для эллиптических задач. М.: Мир, 2013. 512 с.
10. Соболев С. Л. Некоторые применения функционального анализа в математической физике. М.: Наука, 2019. 272 с.